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LETTER TO THE EDITOR

Gauge theory description of spin ladders

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Abstract. An $s = \frac{1}{2}$ antiferromagnetic spin chain is equivalent to the two-flavour massless Schwinger model in a uniform background charge density in the strong coupling regime. The gapless mode of the spin chain is represented by a massless boson of the Schwinger model. In a two-leg spin ladder system the massless boson aquires a finite mass due to inter-chain interactions. The gap energy is found to be about $0.36|J'|$ when the inter-chain Heisenberg coupling J' is small compared with the intra-chain Heisenberg coupling. It is also shown that a cyclically symmetric N_ℓ-leg ladder system is gapless or gapful for an odd or even N_ℓ, respectively.

An $s = \frac{1}{2}$ spin chain with antiferromagnetic nearest-neighbour Heisenberg couplings is exactly solved by the Bethe ansatz [1] and has a gapless excitation. A two-leg spin ladder consists of two spin chains coupled to each other. Experimentally a two-leg spin ladder system has no gapless excitation [2–5]. The gapless mode of spin chains does become gapful. In this paper we give, without resorting to numerical evaluation, a deductive microscopic argument which shows why and how this happens.

Spin ladder systems are not exactly solvable and various approximation methods have been employed in the literature [6–11]. We first show that an $s = \frac{1}{2}$ spin chain is equivalent to the two-flavour massless Schwinger model in the strong coupling regime in a uniform background charge density. The two-flavour Schwinger model has a massless boson excitation, which corresponds to the gapless excitation in the Bethe ansatz. A spin ladder system is described as two sets of two-flavour Schwinger models which interact with each other by four-fermi interactions.

An antiferromagnetic spin chain is described by

$$H_{\text{chain}}(\boldsymbol{S}) = J \sum \boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1} \qquad (J > 0) \tag{1}$$

whereas a two-leg spin ladder is described by

$$\begin{aligned} &H_{\text{ladder}}(\boldsymbol{S}, \boldsymbol{T}) = H_{\text{chain}}(\boldsymbol{S}) + H_{\text{chain}}(\boldsymbol{T}) + H_{\text{rung}}(\boldsymbol{S}, \boldsymbol{T}) \\ &H_{\text{rung}}(\boldsymbol{S}, \boldsymbol{T}) = J' \sum \boldsymbol{S}_n \cdot \boldsymbol{T}_n. \end{aligned} \tag{2}$$

Consider first an $s = \frac{1}{2}$ antiferromagnetic spin chain $H_{\text{chain}}(\boldsymbol{S})$. We express the spin operator in terms of electron operators by $\boldsymbol{S}_n = c_n^\dagger \frac{1}{2} \boldsymbol{\sigma} c_n$. With the aid of the Fierz transformation we have $4\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1} = -\{c_n^\dagger c_{n+1}, c_{n+1}^\dagger c_n\} + 1 - (c_n^\dagger c_n - 1)(c_{n+1}^\dagger c_{n+1} - 1)$. We can drop the last term, as the half-filling condition $c_n^\dagger c_n = 1$ is satisfied for a spin chain. The first term is

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linearized by introducing an auxiriary field, or by the Hubbard–Stratonovich transformation. The Hamiltonian $H_{\rm chain}$ is equivalent to the Lagrangian

$$L^1_{\rm chain} = \sum \left\{ {\rm i}\hbar c_n^\dagger \dot{c}_n - \lambda_n (c_n^\dagger c_n - 1) - \frac{J}{2}(U_n^* U_n - U_n c_n^\dagger c_{n+1} - U_n^* c_{n+1}^\dagger c_n) \right\}. \tag{3}$$

U_n is a link variable, defined on the link connecting sites n and $n+1$. λ_n is a Lagrange multiplier enforcing the half-filling condition at each site. The transformation is valid for $J > 0$. The Lagrangian $L^1_{\rm spin}$ has local $U(1)$ gauge invariance.

We consider a periodic chain of N sites: $\boldsymbol{S}_{N+1} = \boldsymbol{S}_1$. The mean-field energy is evaluated, supposing $|U_n| = U$, to be $E_{\rm mean} = J\{\frac{1}{2}NU^2 - U\cot(\pi/N) + \frac{1}{4}N\}$. For large N it has a sharp minimum at $U = 1/\pi$. Radial fluctuations of U_n's are suppressed, though quantum fluctuations of the phase of U_n's cannot be neglected. We write

$$U_n = \frac{1}{\pi}{\rm e}^{{\rm i}\ell A_n} \tag{4}$$

where ℓ is the lattice spacing. We need to incorporate quantum fluctuations of λ_n and A_n to all orders. With (4) substituted the Lagrangian (3) becomes that of lattice electrodynamics.

To make this point clearer, we take the continuum limit. For an antiferromagnetic spin chain, two sites form one block. The even–odd site index becomes an internal (spin) degree of the Dirac field in the continuum limit. The correspondence is given by

$$\begin{cases} \psi_1^{(a)}(x) = \dfrac{(-{\rm i})^{2s-1}}{\sqrt{2\ell}} c_{2s-1,a} & \text{at odd site} \\ \psi_2^{(a)}(x) = \dfrac{(-{\rm i})^{2s}}{\sqrt{2\ell}} c_{2s,a} & \text{at even site} \end{cases} \tag{5}$$

where x corresponds to $(2s-1)\ell$ and $2s\ell$. With the given normalization $\{\psi_j^{(a)}(x), \psi_k^{(b)}(y)^\dagger\} = \delta^{ab}\delta_{jk}\delta_L(x-y)$ in the continuum limit, where $\delta_L(x)$ is the periodic delta function with the period $L = N\ell$. The phase factors in (5) reflect the Fermi momentum $k_F = \pm\frac{1}{2}\pi$ at the half filling.

The term $\sum_n c_n^\dagger c_{n+1}$ + hermitian conjugate (h.c.) becomes $2{\rm i}\ell \sum_a \int {\rm d}x\, (\psi_1^{(a)\dagger}\partial_x\psi_2^{(a)} + \psi_2^{(a)\dagger}\partial_x\psi_1^{(a)})$. Hence in the continuum limit the original spin Hamiltonian (1) is transformed to a system with the Lagrangian density

$$\mathcal{L}^2_{\rm chain}[A_\mu, \psi] = -\frac{1}{4e^2}F_{\mu\nu}^2 + \sum_{a=1}^{2} c\bar\psi^{(a)}\gamma^\mu \left({\rm i}\hbar\partial_\mu - \frac{1}{c}A_\mu\right)\psi^{(a)} + \frac{1}{\ell}A_0 - \frac{JN}{2\pi^2}. \tag{6}$$

Here the Dirac matrices are $\gamma^0 = \sigma_3, \gamma^1 = {\rm i}\sigma_2$. The 'light' velocity c is given by $c = \ell J/\pi\hbar$. $x_0 = ct$ and $(A_0, A_1) = (\lambda, cA)$. Although the Maxwell term is absent in the $\ell \to 0$ limit, it is generated at finite ℓ. The coupling constant e must be expressed in terms of J and ℓ. From the dimensional analysis $e^2 = k^2 J/\ell$ where k is a constant of O(1). Note that in the $\ell \to 0$ limit with c kept fixed, e^2 diverges as ℓ^{-2}.

This is nothing but the two-flavour massless Schwinger model in the strong coupling regime in a uniform background charge density. The term A_0/ℓ representing the background charge arises from the half-filling condition. The system is neutral as a whole.

Note that the spin index a of original electrons becomes a flavour index in (6), while the even–odd index j becomes a spin index of the Dirac field $\psi^{(a)}(x)$. The two-flavour nature reflects the electron spin $\frac{1}{2}$.

The correspondence of the spin chain model to quantum electrodynamics (QED_2) has been noted in the literature, but the rigorous derivation has not been given before [12]. In particular, the importance of the two-flavour nature has not been recognized. Mapping to $SU(2)$ gauge theory has also been suggested [13].

The two-flavour massless Schwinger model is exactly solvable [14]. Quantum fluctuations of all fields, $\psi^{(a)}$ and A_μ, can be completely taken into account. With the periodic boundary condition, the model is two-flavour QED_2 defined on a circle, which has been analysed in detail by the bosonization method [15–17].

The bosonization formula for the left- and right-moving components of the Dirac fields is

$$\psi^a_\pm(t,x) = \frac{1}{\sqrt{L}} C^a_\pm \mathrm{e}^{\pm\mathrm{i}\{q^a_\pm + 2\pi p^a_\pm (t\pm x)/L\}} N_0[\mathrm{e}^{\pm\mathrm{i}\sqrt{4\pi}\phi^a_\pm(t,x)}] \qquad (a=1,2) \tag{7}$$

where $C^a_+ = \mathrm{e}^{\mathrm{i}\pi\sum_{b=1}^{a-1}(p^b_+ + p^b_-)}$ and $C^a_- = \mathrm{e}^{\mathrm{i}\pi\sum_{b=1}^{a}(p^b_+ - p^b_-)}$. ϕ^a_+ (ϕ^a_-) represents left- (right-) moving modes. $N_0[]$ denotes the normal ordering in a basis of massless fields. The Hamiltonian becomes [17]

$$\begin{aligned} H^2_{\mathrm{chain}} &= \frac{e^2L}{2}P_W^2 + \sum_{a=1}^{2}\frac{\pi\hbar c}{2L}\left\{Q_a^2 + \left(Q_{5a} + \frac{\Theta_W}{\pi}\right)^2\right\} \\ &\quad + \int_0^L \mathrm{d}x\,\frac{\hbar c}{2}\left(\frac{1}{c^2}\dot{\Phi}^2 + \Phi'^2 + \frac{2e^2}{\pi\hbar c}\Phi^2 + \frac{1}{c^2}\dot{\chi}^2 + \chi'^2\right). \end{aligned} \tag{8}$$

The neutrality condition reads $Q_1 + Q_2 = L/\ell = N$. Θ_W and P_W are the Wilson line phase $\mathrm{e}^{\mathrm{i}\Theta_W} = \exp[(\mathrm{i}/\hbar c)\int_0^L \mathrm{d}x\, A_1]$ and its conjugate momentum. $Q_a = -p_a^+ + p_a^-$ and $Q_{5a} = p_a^+ + p_a^-$ are charge and axial charge of the ath flavour, respectively, both of which take integer eigenvalues and commute with the Hamiltonian. $\Phi = (\phi_1 + \phi_2)/\sqrt{2}$ and $\chi = (\phi_1 - \phi_2)/\sqrt{2}$ where $\phi_a = \phi^a_+ + \phi^a_-$ and $\int \mathrm{d}x\,\phi_a = 0$.

The Φ field has a Schwinger mass μ where $\mu^2 = 2e^2\hbar/\pi c^3$. The excitation energy is $\mu c^2 = \sqrt{2}kJ/\pi \sim 0.45kJ$. The χ field is massless, which corresponds to the gapless excitation in the spin chain and controls the behaviour of correlation functions at large distances. The wavefunction for the zero mode part is written as

$$\begin{aligned} |\Psi\rangle &= \sum_{n,r}\int \mathrm{d}p_W\, |p_W, n, r\rangle \mathrm{e}^{-\mathrm{i}r\varphi + 2\pi\mathrm{i}np_W} f(p_W, \varphi + \pi p_W) \\ P_W|p_W,n,r\rangle &= p_W|p_W,n,r\rangle \\ p^a_\pm|p_W,n,r\rangle &= (n + r\delta_{a,1} \mp \tfrac{1}{4}N)|p_W,n,r\rangle \end{aligned} \tag{9}$$

where $f(p_W,\varphi)$ must solve the Schrödinger equation

$$\begin{aligned} K(p_W,\varphi)f(p_W,\varphi) &= \epsilon f(p_W,\varphi) \\ K(p_W,\varphi) &= -\frac{1}{\pi^2}\frac{\partial^2}{\partial p_W^2} - \frac{\partial^2}{\partial\varphi^2} - \left(\frac{\mu c L p_W}{2\hbar}\right)^2. \end{aligned} \tag{10}$$

For the ground state $f(p_W,\varphi) = \text{constant}\ \mathrm{e}^{-\pi\mu c L p_W^2/4\hbar}$.

In the Schwinger model there is a θ parameter characterizing states. The wavefunction (9) corresponds to $\theta = 0$. The θ vacuum originates from the invariance under large gauge transformatins and the chiral anomaly in the continuum theory [16]. In the lattice spin systems the lowest energy state with $\theta = 0$ is expected to be singled out.

Employing the bosonization formula, the critical exponent of the spin–spin correlation function $\langle \boldsymbol{S}(2n)\boldsymbol{S}(0)\rangle \sim n^{-\eta}$ ($n \gg 1$, $n \ll N$) is found to be $\eta = 1$, which agrees with the result from the Bethe ansatz.

Now we consider a spin ladder system (2). In the absence of the inter-chain rung interaction (J'=0) the system is equivalent to the two sets of two-flavour massless Schwinger models described by $\mathcal{L}^2_{\rm chain}[A_\mu, \psi] + \mathcal{L}^2_{\rm chain}[\tilde{A}_\mu, \tilde{\psi}]$. With the aid of the correspondence (5), the inter-chain interaction $H_{\rm rung}$ in the continuum limit is written as

$$H^3_{\rm rung} = \frac{J'N}{2} + \frac{\ell J'}{4} \int {\rm d}x \, (\{\psi^\dagger\tilde{\psi}, \tilde{\psi}^\dagger\psi\} + \psi^\dagger\psi \cdot \tilde{\psi}^\dagger\tilde{\psi} + \{\bar{\psi}\tilde{\psi}, \bar{\tilde{\psi}}\psi\} + \bar{\psi}\psi \cdot \bar{\tilde{\psi}}\tilde{\psi}) \qquad (11)$$

where every quantity in the expression is a flavour singlet; $\psi^\dagger\tilde{\psi} = \sum_{a=1}^2 \psi^{(a)\dagger}\tilde{\psi}^{(a)}$ etc. Note that both charge and scalar density operators appear in (11). The chiral symmetry is broken, which leads to mass generation.

When expressed in terms of $\psi_\pm$ and $\tilde{\psi}_\pm$, $H^3_{\rm rung}$ contains many terms. The Hamiltonian is simplified in the large volume limit $L = N\ell \to \infty$. We define $\rho_a = \psi^{(a)\dagger}\psi^{(a)}$, $M_a = \psi_+^{(a)\dagger}\psi_-^{(a)}$, and the corresponding $\tilde{\rho}_a$ and $\tilde{M}_a$. The relevant terms in $H^3_{\rm rung}$ are

$$\begin{aligned} &H^3_{\rm rung} \sim H_{3a} + H_{3b} \\ &H_{3a} = \frac{J'\ell}{4} \int {\rm d}x \, (\rho_1 - \rho_2)(\tilde{\rho}_1 - \tilde{\rho}_2) \\ &H_{3b} = \frac{J'\ell}{4} \int {\rm d}x \, \{(M_1 - M_2)(\tilde{M}_1^\dagger - \tilde{M}_2^\dagger) + ({\rm h.c.})\}. \end{aligned} \qquad (12)$$

Terms of the form $M_a\tilde{M}_b$ are suppressed as fluctuations in Q_a are small compared with the average $N/2$.

Boson fields associated with ψ and $\tilde{\psi}$ are denoted by (Φ, χ) and $(\tilde{\Phi}, \tilde{\chi})$, respectively. We introduce a new orthonormal basis: $\Phi_\pm = (\Phi \pm \tilde{\Phi})/\sqrt{2}$ and $\chi_\pm = (\chi \pm \tilde{\chi})/\sqrt{2}$. The first term in (12) is

$$H_{3a} = \frac{J'\ell}{4L}(Q_1 - Q_2)(\tilde{Q}_1 - \tilde{Q}_2) + \int {\rm d}x \, \frac{J'\ell}{4\pi}\{(\partial_x\chi_+)^2 - (\partial_x\chi_-)^2\}. \qquad (13)$$

It changes the propagation velocities of $\chi_\pm$ fields.

It follows from (7) that

$$M_a\tilde{M}_b^\dagger = {\rm e}^{-2\pi{\rm i}(Q_a - \tilde{Q}_b)x/L}{\rm e}^{-{\rm i}(q_a - \tilde{q}_b)}\frac{1}{L^2}N_0[{\rm e}^{-{\rm i}\sqrt{4\pi}(\phi_a - \tilde{\phi}_b)}]. \qquad (14)$$

Note that $N_0[{\rm e}^{{\rm i}\beta\chi}] = B(mcL/\hbar)^{\beta^2/4\pi}N_m[{\rm e}^{{\rm i}\beta\chi}]$ where the reference mass in the normal ordering $N[\ ]$ is shifted from 0 to m. $B(0)$=1 and $B(z) \sim e^\gamma z/4\pi$ for $z \gg 1$ [16]. That is, if all fields become massive, (14) is nonvanishing in the $L \to \infty$ limit. Otherwise (14) vanishes. In passing, terms not included in (12) are suppressed exponentially in the $L \to \infty$ limit when $\chi_\pm$ fields aquire masses.

There are fluctuations in Q_a. Write $Q_{1,2} = \frac{1}{2}N \pm Q$ and $\tilde{Q}_{1,2} = \frac{1}{2}N \pm \tilde{Q}$. Important terms in $\int {\rm d}x \, M_a\tilde{M}_b^\dagger$ result when $Q = \pm\tilde{Q}$. Since $|Q|, |\tilde{Q}| \ll N$, we have in the large volume limit

$$\begin{aligned} H_{3b} = \frac{J'\ell}{4}\left(\frac{e^\gamma c}{4\pi\hbar}\right)^2 \int {\rm d}x \, [&\mu_{\Phi_-}\mu_{\chi_-}\{{\rm e}^{-{\rm i}(q_1 - \tilde{q}_1)}N[{\rm e}^{-{\rm i}\sqrt{4\pi}(\Phi_- + \chi_-)}] \\ &+{\rm e}^{-{\rm i}(q_2 - \tilde{q}_2)}N[{\rm e}^{-{\rm i}\sqrt{4\pi}(\Phi_- - \chi_-)}]\} \\ &-\mu_{\Phi_-}\mu_{\chi_+}\{{\rm e}^{-{\rm i}(q_1 - \tilde{q}_2)}N[{\rm e}^{-{\rm i}\sqrt{4\pi}(\Phi_- + \chi_+)}] + {\rm e}^{-{\rm i}(q_2 - \tilde{q}_1)}N[{\rm e}^{-{\rm i}\sqrt{4\pi}(\Phi_- - \chi_+)}]\} + {\rm h.c.}]. \end{aligned} \qquad (15)$$

Here we have defined $\Phi_\pm = (\Phi \pm \tilde{\Phi})/\sqrt{2}$ and $\chi_\pm = (\chi \pm \tilde{\chi})/\sqrt{2}$. $N[\mathrm{e}^{-\mathrm{i}\sqrt{4\pi}(\Phi_- + \chi_-)}]$ denotes that the Φ_- and χ_- fields are normal-ordered with respect to their masses μ_{Φ_-} and μ_{χ_-}, respectively.

H_{3b} has two major effects. It gives an additional potential in the zero mode sector:

$$\Delta H_{\mathrm{zero}} = L\frac{J'\ell}{4}\left(\frac{e^\gamma c}{4\pi\hbar}\right)^2 \mu_{\Phi_-}\{\mu_{\chi_-}[\mathrm{e}^{-\mathrm{i}(q_1-\tilde{q}_1)} + \mathrm{e}^{-\mathrm{i}(q_2-\tilde{q}_2)}] - \mu_{\chi_+}[\mathrm{e}^{-\mathrm{i}(q_1-\tilde{q}_2)} + \mathrm{e}^{-\mathrm{i}(q_2-\tilde{q}_1)}] + (\mathrm{h.c.})\}. \tag{16}$$

Secondly it gives additional masses to Φ_- and $\chi_\pm$. For small $|J'| \ll J$

$$\begin{aligned}
\mu^2_{\Phi_-} &= \mu^2 - \frac{e^{2\gamma}}{4\pi}\frac{J'\ell}{\hbar c}\mu_{\Phi_-}(\mu_{\chi_-}\langle \mathrm{e}^{\pm\mathrm{i}(q_1-\tilde{q}_1)}\rangle - \mu_{\chi_+}\langle \mathrm{e}^{\pm\mathrm{i}(q_1-\tilde{q}_2)}\rangle)\\
\mu^2_{\chi_-} &= -\frac{e^{2\gamma}}{4\pi}\frac{J'\ell}{\hbar c}\mu_{\Phi_-}\mu_{\chi_-}\langle \mathrm{e}^{\pm\mathrm{i}(q_1-\tilde{q}_1)}\rangle\\
\mu^2_{\chi_+} &= \frac{e^{2\gamma}}{4\pi}\frac{J'\ell}{\hbar c}\mu_{\Phi_-}\mu_{\chi_+}\langle \mathrm{e}^{\pm\mathrm{i}(q_1-\tilde{q}_2)}\rangle
\end{aligned} \tag{17}$$

Here we have made use of $\langle \mathrm{e}^{\pm\mathrm{i}(q_1-\tilde{q}_1)}\rangle = \langle \mathrm{e}^{\pm\mathrm{i}(q_2-\tilde{q}_2)}\rangle$ and $\langle \mathrm{e}^{\pm\mathrm{i}(q_1-\tilde{q}_2)}\rangle = \langle \mathrm{e}^{\pm\mathrm{i}(q_2-\tilde{q}_1)}\rangle$, which reflects the up–down symmetry of the original spin system and is justified shortly.

The wavefunction of the ladder system is specified with $f(p_W, \varphi; \tilde{p}_W, \tilde{\varphi})$ as in (9). The rung interaction (16) gives an additional potential in the φ representation. $\mathrm{e}^{\mathrm{i}q_1}$ and $\mathrm{e}^{\mathrm{i}q_2}$ give rise to $\mathrm{e}^{\mathrm{i}\varphi - \mathrm{i}\pi p_W}$ and $\mathrm{e}^{-\mathrm{i}\varphi - \mathrm{i}\pi p_W}$, respectively. f satisfies

$$\begin{aligned}
&\{K(p_W,\varphi) + K(\tilde{p}_W,\tilde{\varphi}) + V_{\mathrm{rung}}\}f = \epsilon f\\
&V_{\mathrm{rung}} = L^2\frac{J'\ell}{\pi\hbar c}\left(\frac{e^\gamma c}{4\pi\hbar}\right)^2\mu_{\Phi_-}\{\mu_{\chi_-}\cos(\varphi-\tilde{\varphi}) - \mu_{\chi_+}\cos(\varphi+\tilde{\varphi})\}\cos\pi(p_W - \tilde{p}_W).
\end{aligned} \tag{18}$$

For large L the potential term dominates in equation (18). The ground-state wavefunction has a sharp peak at the minimum of the potential. For $J' > 0$ ($J' < 0$), the minimum occurs at $p_W = \tilde{p}_W = 0$ and $\varphi = -\tilde{\varphi} = \pm\frac{1}{2}\pi$ ($\varphi = \tilde{\varphi} = \pm\frac{1}{2}\pi$) so that

$$\langle \mathrm{e}^{\pm\mathrm{i}(q_a-\tilde{q}_a)}\rangle = -\langle \mathrm{e}^{\pm\mathrm{i}(q_1-\tilde{q}_2)}\rangle = -\langle \mathrm{e}^{\pm\mathrm{i}(q_2-\tilde{q}_1)}\rangle = \mp 1 \qquad \text{for } \begin{cases} J' > 0\\ J' < 0.\end{cases} \tag{19}$$

The masses are determined by (17) and (19):

$$\begin{aligned}
&\mu_{\Phi_-} = \frac{\mu}{\sqrt{1-2\kappa^2}} \qquad \mu_{\chi_-} = \mu_{\chi_+} = \frac{\kappa\mu}{\sqrt{1-2\kappa^2}}\\
&\kappa = \frac{e^{2\gamma}}{4\pi}\frac{|J'|\ell}{\hbar c} = \frac{e^{2\gamma}}{4}\frac{|J'|}{J} \sim 0.79\frac{|J'|}{J}.
\end{aligned} \tag{20}$$

The expression is valid for small κ. The excitation energy, a spin gap, is

$$\Delta_{\mathrm{spin}} = \mu_{\chi_\pm}c^2 \sim \kappa\mu c^2 = \frac{e^{2\gamma}k}{2^{3/2}\pi}|J'| = 0.36k|J'|. \tag{21}$$

The ratio of Δ_{spin} to μc^2 is κ. The gapless mode becomes gapful. The spin gap is determined by $|J'|$, generated irrespective of the sign of J'. The energy density is lowered:

$$\Delta\mathcal{E} = -\frac{\Delta^2_{\mathrm{spin}}}{2\ell J}. \tag{22}$$

We have shown that the rung interaction breaks the chiral symmetry of spin chain systems, and generates a spin gap.

In the literature the spin gap has been determined by various numerical methods for varying J'/J [8]. In particular, Greven *et al* obtained $\Delta_{\rm spin} = 0.41J'$ for small J'/J and $0.50J'$ for $J' = J$, which is consistent with our prediction (21).

It has been well known that spin chain systems are mapped to nonlinear sigma models [18]. Sierra has applied this mapping to N_ℓ-leg ladder systems of spin S, and has shown that the spectrum is gapful or gapless for an integer or half-odd-integer SN_ℓ, respectively [9]. The mapping to sigma models is valid for large $SN_\ell \gg 1$, while our method of mapping to the Schwinger model works for $S = \frac{1}{2}$.

The method of bosonization has been employed in the spin ladder problem. Schulz, in analysing a spin S chain, expressed $\boldsymbol{S}$ as a sum of $2S$ spin-$\frac{1}{2}$ vectors, thereby transforming the spin chain to a special kind of a spin-$\frac{1}{2}$ ladder system. With the aid of bosonization and renormalization group analysis he concluded that the spectrum is gapless for a half-odd-integer S.[10]

More recently a 2-leg $s = \frac{1}{2}$ ladder system has been analysed by bosonization by Shelton *et al* and by Kishine and Fukuyama [11]. They have obtained a similar Hamiltonian to ours, but could not determined the gap. Our bosonization formula (7) is a rigorous operator identity with no ambiguity in normalization, with which the Hamiltonian is transformed in the bosonized form. The correct treatment of the normal ordering is crucial in dealing with the mass (gap) generation. Not only the light modes ($\chi_\pm$) but also the heavy modes ($\Phi_\pm$) and zero modes (Θ, q_a) play an important role, which has been dismissed in [11].

Our argument can be generalized to N_ℓ-leg $s = \frac{1}{2}$ ladder systems. Inter-chain interactions are given by $H_{\rm rung} = \sum_{(ij)} J'_{ij} \sum_n \boldsymbol{S}_n^{(i)} \boldsymbol{S}_n^{(j)}$ where i and j are chain indices and (ij) labels rung pairs. $J'_{ij} = 2J$ for all (ij) in Schulz' model in [10].

Let us consider a cyclically symmetric antiferromagnetic ladder system in which nonvanishing J'_{ij}'s are $J'_{i,i+1} = J' > 0$ where $J'_{N_\ell,N_\ell+1} \equiv J'_{N_\ell,1}$. Among boson fields Φ_i's or χ_i's, the singlet combination is denoted by Φ_+ or χ_+. Other combinations of Φ's or χ's are degenerate. There are four masses to be determined: $\mu_{\Phi_\pm}$ and $\mu_{\chi_\pm}$. $\mu_{\Phi_\pm} \sim \mu$ for small $|J'|$. The issue is whether or not all χ fields become massive. The crucial part is the mass of χ_+.

Repeating the above argument, one finds that the part of the rung potential $V_{\rm rung}$ in (18), $\mu_{\chi_-} \cos(\varphi - \tilde{\varphi}) - \mu_{\chi_+} \cos(\varphi + \tilde{\varphi})$, is replaced by

$$\mu_{\chi_-} \sum_{i=1}^{N_\ell} \cos(\varphi_i - \varphi_{i+1}) - \mu_{\chi_+}^{2/N_\ell} \mu_{\chi_-}^{1-(2/N_\ell)} \sum_{i=1}^{N_\ell} \cos(\varphi_i + \varphi_{i+1}) \tag{23}$$

where $\varphi_{N_\ell+1} = \varphi_1$. If $\mu_{\chi_-} = 0$, $V_{\rm rung} = 0$ and no correction arises for $\mu_{\Phi_\pm}$ or $\mu_{\chi_\pm}$. This solution has a higher energy density than the non-trivial solution so that $\mu_{\chi_-} \neq 0$. From the symmetry $V_{\rm rung}$ is minimized at $\cos(\varphi_i - \varphi_{i+1}) = f_-$ ($i = 1, \ldots, N_\ell$). This implies that $\varphi_j = \varphi + (j-1)\eta$ and $\eta = 2p\pi/N_\ell$ or $2p\pi/(N_\ell - 2)$ where p is an integer.

Suppose $\mu_{\chi_+} \neq 0$. Then $\cos(\varphi_i + \varphi_{i+1}) = f_+$ ($i = 1, \ldots, N_\ell$). This leads to an additional condition that $\eta = \pi$. All of these conditions are satisfied for an even N_ℓ. The potential is minimized at $\varphi_{2p+1} = \pm\frac{1}{2}\pi$ and $\varphi_{2p} = \mp\frac{1}{2}\pi$. For an odd N_ℓ the conditions cannot be satisfied.

If $\mu_{\chi_+} = 0$, η need not be π. This gives a solution for an odd N_ℓ. For an even N_ℓ, this solution yields a higher energy density than the solution with $\mu_{\chi_+} \neq 0$ above. To summarize, the spectrum is gapless for an odd N_ℓ, but is gapful for an even N_ℓ. The interaction is frustrated in the rung direction for an odd N_ℓ. The argument here is similar to Schulz' in [10].

In the experimental samples [4] $J' \sim J$ so that $\kappa = {\rm O}(1)$. For instance, in $\rm SrCu_2O_3$ (2-

leg ladder), $J \sim J' \sim 1300$ K and $\Delta_{\rm spin} \sim 420$ K. The formula (17) need to be improved by taking account of effects of the nonlinear terms in (15). Further, it is observed that spin ladder systems with three legs are gapless. (The experimental sample is not cyclically symmetric: $J'_{12} = J'_{23} \sim J$ but $J'_{13} = 0$.) For this the large value of κ is important, as our analysis indicates that a gap is generated so long as κ is sufficiently small. It has also been reported that the spin gap is not affected by nonmagnetic impurities [5]. We will come back to these points in separate publications.

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